

# Some developments in vertex operator algebra theory, old and new

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## Abstract

In this exposition, I discuss several developments in the theory of vertex operator algebras, and I include motivation for the definition.

This is a more detailed version of the talk I gave at the Conference on Lie Algebras, Vertex Operator Algebras and Their Applications. I thank Yi-Zhi Huang and Kailash Misra very much for organizing this conference.

I would like to motivate the concept of vertex (operator) algebra, including the definition; to discuss some of the main sources of the theory, including Lie algebras and partition identities, the “monstrous moonshine” problem, and string theory and conformal field theory; and to mention a selection of developments. (This writeup is not intended to be a comprehensive survey.) Some of this talk is drawn from the introductory material in [FLM3]–[FLM5], [DL2] and [LL], as well as from earlier expositions such as [LW5], [L1], [L2], [HL7], [HL8] and [L4].

The mathematical notion of “vertex algebra” was introduced by R. Borcherds [B1]. A variant of it, a notion of “vertex operator algebra,” was introduced in [FLM5]. These notions are algebraic formulations of concepts that had been developed by many string theorists, conformal field theorists and quantum field theorists, and formalized in [BPZ] as certain “operator algebras,” later called “chiral algebras” in physics.

Vertex (operator) algebra theory is inherently “nonclassical,” in the same spirit in which string theory in physics is nonclassical and also in the same spirit in which the sporadic finite simple groups in mathematics are nonclassical. String theory, its initial version having been introduced in the late 1960s, is based on the premise that elementary particles manifest themselves as “vibrational modes” of fundamental strings, rather than points, moving through space according to quantum-field-theoretic principles. A string sweeps out a two-dimensional “world-sheet” in space-time, and it is fruitful to focus on the case in which the surface is a Riemann surface locally parametrized by a complex coordinate. The resulting two-dimensional conformal (quantum) field theory has been studied extensively.

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Meanwhile, in mathematics, the vast program to classify the finite simple groups (see [Gor]) had reached a dramatic point around 1980. What turned out to be the largest finite simple group that is sporadic (i.e., not belonging to one of the infinite families), the Fischer-Griess Monster  $\mathbb{M}$ , had been predicted by B. Fischer and R. Griess and was constructed by Griess [G] as a symmetry group (of order about  $10^{54}$ ) of a remarkable new commutative but very, very highly nonassociative, seemingly ad-hoc, algebra  $\mathbb{B}$  of dimension 196,883. The “structure constants” of the Griess algebra  $\mathbb{B}$  were “forced” by expected properties of the conjectured-to-exist Monster. It was proved by J. Tits that  $\mathbb{M}$  is actually the full symmetry group of  $\mathbb{B}$ .

A bit earlier (1978–79), J. McKay, J. Thompson, J. Conway and S. Norton (see especially Conway-Norton [CN]) had discovered astounding “numerology” culminating in the “monstrous moonshine” conjectures relating the not-yet-proved-to-exist Monster  $\mathbb{M}$  to modular functions in number theory, namely:

There should exist a (natural) infinite-dimensional  $\mathbb{Z}$ -graded module for  $\mathbb{M}$  (i.e., representation of  $\mathbb{M}$ )

$$V = \bigoplus_{n=-1,0,1,2,3,\dots} V_n$$

such that

$$\sum_{n=-1,0,1,2,3,\dots} (\dim V_n) q^n = J(q), \quad (1)$$

where

$$J(q) = q^{-1} + 0 + 196884q + \text{higher-order terms}, \quad (2)$$

the classical modular function with its constant term set to 0.  $J(q)$  is the suitably normalized generator of the field of  $SL(2, \mathbb{Z})$ -modular invariant functions on the upper half-plane, with  $q = e^{2\pi i \tau}$ ,  $\tau$  in the upper half-plane. (Note:  $196884=196883+1$ .) The existence of such a structure  $V$  was conjectured by McKay and Thompson.

More generally, as Conway and Norton conjectured, for every  $g \in \mathbb{M}$  (not just  $g = 1$ ), the generating function

$$\sum_{n=-1,0,1,2,3,\dots} (\text{tr } g|_{V_n}) q^n \quad (3)$$

should be the analogous “Hauptmodul” for a suitable discrete subgroup of  $SL(2, \mathbb{R})$ , a subgroup having a fundamental “genus-zero property,” so that its associated field of modular-invariant functions has a single generator (a Hauptmodul). (The left-hand side of (1) is the *graded dimension* of the graded vector space  $V$ , and (3) is the *graded trace* of the action of  $g$  on the graded space  $V$ ; the graded dimension is of course the graded trace of the identity element  $g = 1$ .) The Conway-Norton conjecture subsumed a remarkable coincidence that had been noticed earlier—that the 15 primes giving rise to the genus-zero property (see A. Ogg [O]) are precisely the primes dividing the order of the (conjectured-to-exist) Monster.

Proving these conjectures would give a remarkable connection between classical number theory and “nonclassical” sporadic group theory. The existence of a structure  $V$  was soon essentially (but nonconstructively) proved by Thompson, A. O. L. Atkin, P. Fong and S. Smith. After Griess constructed  $\mathbb{M}$ , with I. Frenkel and A. Meurman [FLM2] we (constructively) proved the McKay-Thompson conjecture, that there should exist a natural (whatever that was going to mean) infinite-dimensional  $\mathbb{Z}$ -graded  $\mathbb{M}$ -module  $V$  whose graded dimension is  $J(q)$ , as in (1). The graded traces of some, but not all, of the elements of the Monster—the elements of an important subgroup of  $\mathbb{M}$ , namely, a certain involution centralizer involving the largest Conway sporadic group  $Co_1$ —were consequences of the construction, and these graded traces were indeed (suitably) modular functions [FLM2]. We called this  $V$  “the moonshine module  $V^\natural$ ” because of its naturality (although the construction of  $V^\natural$  is not short). See J. Tits ([Ti1], [Ti2]) for discussions of the construction of the Monster and of the moonshine module.

The construction [FLM2] heavily used a number of different types of “vertex operators,” all of them recently constructed, or newly constructed as steps in [FLM2], along with their algebraic structure and relations. These were needed for the construction of the structure  $V^\natural$  itself, including a natural “algebra of vertex operators” acting on it. They were also needed for the construction of a natural infinite-dimensional “affinization” of the Griess algebra  $\mathbb{B}$  acting on  $V^\natural$ . This “affinization,” which was part of the new algebra of vertex operators, is analogous to, but more subtle than, the notion of affine Lie algebra, an example of which is discussed below. More precisely, the vertex operators were needed for a “commutative affinization” of a certain natural 196884-dimensional enlargement  $\mathcal{B}$  of  $\mathbb{B}$ , with an identity element (rather than a “zero” element) adjoined to  $\mathbb{B}$ . This enlargement  $\mathcal{B}$  naturally incorporated the Virasoro algebra—the central extension of the Lie algebra of formal vector fields on the circle—acting on  $V^\natural$ ; for us, the Virasoro algebra arose not because of its role as a fundamental “symmetry algebra” in string theory but rather because of the fact that a natural identity element for the algebra was “forced” on us by our construction. The vertex operators were also needed for a natural “lifting” of Griess’s action of  $\mathbb{M}$  from the finite-dimensional space  $\mathbb{B}$  to the infinite-dimensional structure  $V^\natural$ , including its algebra of vertex operators and its copy of the affinization of  $\mathcal{B}$ . Thus the Monster was now realized as the symmetry group of a certain explicit “algebra of vertex operators” based on an infinite-dimensional  $\mathbb{Z}$ -graded structure whose graded dimension is the modular function  $J(q)$ .

Griess’s construction of  $\mathbb{B}$  and of  $\mathbb{M}$  acting on  $\mathbb{B}$  was a crucial guide for us, although we did not start by using his construction; rather, we recovered it, as a finite-dimensional “slice” of a new infinite-dimensional construction using vertex operator considerations. In fact, our presentation of the Griess algebra (see [FLM1], [FLM2], [FLM5]) was short and entirely canonical, and involved no choices or guesses of signs or structure constants; one “reads” this algebra as in [FLM1] from canonical untwisted and twisted vertex operator structures newly constructed starting from the Leech lattice (mentioned below). As presented in [FLM1], the 196884-dimensional algebra is simply the direct sum of

the weight-two subspace of the canonical involution-fixed subspace of the untwisted Leech lattice vertex operator structure with the analogous subspace of a canonically twisted vertex operator structure; the commutative nonassociative algebra structure and natural “associative” symmetric bilinear form structure on  $\mathcal{B}$  are essentially described “in words.” The initially strange-seeming *finite-dimensional* Griess algebra was now embedded in a natural new *infinite-dimensional* space on which a certain algebra of vertex operators acts, via a new kind of “generalized commutation relation” (relations of this type are discussed below); such relations are what gave the commutative affinization mentioned above. At the same time, the Monster, a *finite* group, took on a new appearance by now being understood in terms of a natural *infinite-dimensional* structure. The very-highly-nonassociative Griess algebra, or rather, from our viewpoint, the natural modification of the Griess algebra, with an identity element adjoined, coming from a “forced” copy the Virasoro algebra, became simply the conformal-weight-two subspace of an algebra of vertex operators of a certain “shape.” The word “simply” refers to the ease of defining a commutative nonassociative algebra with an associative symmetric bilinear form (generalizing the Griess algebra with identity element adjoined, for the special case of  $V^\natural$ ) in the new general context of algebras of vertex operators of “shape” similar to that of  $V^\natural$  (as was explained in [FLM2] and [FLM5]); the actual construction of the particular algebra  $V^\natural$  remains complex. In any case, the largest sporadic *finite* simple group, the Monster, was “really” *infinite-dimensional*.

In the expansion (2), the constant term of  $J(q)$  is zero, and this choice of constant term, which is not uniquely determined by number-theoretic principles, is not traditional in number theory. It turned out that the vanishing of the constant term in (2) was canonically “forced” by the requirement that the Monster should act naturally on  $V^\natural$  and on an associated algebra of vertex operators. This vanishing of the degree-zero subspace of  $V^\natural$  is actually analogous in a certain strong sense to the absence of vectors in the Leech lattice of square-length two; the Leech lattice is a distinguished rank-24 even unimodular (self-dual) lattice with no vectors of square-length two. In addition, this vanishing of the degree-zero subspace of  $V^\natural$  and the absence of square-length-two elements of the Leech lattice are in turn analogous to the absence of code-words of weight 4 in the Golay error-correcting code, a distinguished self-dual binary linear code on a 24-element set, with the lengths of all code-words divisible by 4. In fact, the Golay code was used in the original construction of the Leech lattice, and the Leech lattice was used in the construction of  $V^\natural$ . This was actually to be expected (if  $V^\natural$  existed) because it was well known that the automorphism groups of both the Golay code and the Leech lattice are (essentially) sporadic finite simple groups; the automorphism group of the Golay code is the Mathieu group  $M_{24}$  and the automorphism group of the Leech lattice is a double cover of the Conway group  $Co_1$  mentioned above, and both of these sporadic groups were well known to be involved in the Monster (if it existed) in a fundamental way. The work [FLM2], [FLM5] revealed, and exploited, a new hierarchy, namely: error-correcting codes, lattices, and vertex-operator-theoretic structures. The Golay code is actually unique subject to its distinguishing properties mentioned

above (proved by V. Pless [Pl]) and the Leech lattice is unique subject to its distinguishing properties mentioned above (proved by Conway [Co] and others). Is  $V^\natural$  unique? If so, unique subject to what? The answer to this question can be viewed as serving as a motivation of the very notion of vertex operator algebra. But this uniqueness is an unsolved problem; more on this below.

After [FLM2] appeared, what has been called the “first string theory revolution” started in the summer of 1984, stimulated by the work [GS] of M. Green and J. Schwarz. In this suddenly-active period in string theory, the new structure  $V^\natural$  came to be viewed in retrospect by string theorists as an inherently *string-theoretic* structure: the “chiral algebra” underlying the  $\mathbb{Z}_2$ -orbifold conformal field theory based on the Leech lattice. The string-theoretic geometry is this: One takes the torus that is the quotient of 24-dimensional Euclidean space modulo the Leech lattice, and then one takes the quotient of this manifold by the “negation” involution  $x \mapsto -x$ , giving rise to an orbit space called an “orbifold”—a manifold with, in this case, a “conical” singularity. Then one takes the “conformal field theory” (presuming that it exists mathematically) based on this orbifold, and from this one forms a “string theory” in two-dimensional space-time by compactifying a 26-dimensional “bosonic string” on this 24-dimensional orbifold. The string vibrates in a 26-dimensional space, 24 dimensions of which are curled into this 24-dimensional orbifold, and space-time is thus 2-dimensional in this “toy-model” string theory. Such an adjunction of a two-dimensional structure is a natural and standard procedure in string theory; 26 is the “critical dimension” in bosonic string theory. So in retrospect, the mathematical construction [FLM2] was essentially the construction of an orbifold string theory (actually, the first example of a theory of a string propagating on an orbifold that is not a torus). As discussed in the Introduction of [FLM5], some of the basic string-theoretic papers on these aspects of orbifold string theory are [DHVW1], [DHVW2], [Ha], [DFMS], [HV], [NSV], [M], [DGH] and [DVVV]. The idea of “orbifolding” (as string theorists were to call it) came, in the development of the work [FLM2], [FLM5], from the construction of general twisted vertex operators and their algebraic relations, including relations involving what sometimes came to be called “intertwining operators” among “twisted sectors,” treated in detail in [FLM5] and related works discussed there. The construction in [FLM2] also came to be viewed as a conformal-field-theoretic structure in the sense of [BPZ], which appeared around the same time as [FLM2]. (These ideas are all discussed in [FLM5].)

As I mentioned at the beginning, in [B1] Borchers introduced the axiomatic notion of *vertex algebra*. This naturally extended the relations [FLM2] for the vertex operators for  $V^\natural$  and also other known mathematical and physical features of known vertex operators, and these axioms turned out to be essentially equivalent to Belavin-Polyakov-Zamolodchikov’s physical axioms [BPZ] for the basic “algebras” of vertex operators underlying conformal field theory. In [B1] Borchers asserted that  $V^\natural$  admits a vertex algebra structure, generated by the algebraic structure constructed in [FLM2], on which  $\mathbb{M}$  (still) acts as a symmetry group. This assertion was proved in [FLM5], by an (elaborate) extension of the proof of the results announced in [FLM2] (rather than by a direct use of

the results announced in [FLM2]). In [FLM5], Borchers’s definition of vertex algebra was modified, giving the variant notion of *vertex operator algebra*. The main modification was the introduction of what we called the “Jacobi identity,” discussed below. (Actually, we had first thought of this identity as the “master formula” for reasons mentioned below, but then we decided to emphasize its analogy with the Jacobi identity in the definition of the notion of Lie algebra.) Another modification was the emphasis of the viewpoint that the elements of the algebra essentially “are” vertex operators. Also, the definition of “vertex operator algebra” in [FLM5] included two natural grading-restriction conditions and the presence of a copy of the Virasoro algebra, because these features naturally arose in the construction of  $V^\natural$ . The term “vertex algebra” generally refers to any notion equivalent to Borchers’s notion in [B1], and the term “vertex operator algebra” generally refers to any notion equivalent to the notion in [FLM5] and also in the sequel [FHL]—that is, including the two grading restrictions—even though, from the strictly logical point of view, the notion in [FLM5] and [FHL] does not have any more “operators” in it than does the notion in [B1] (except in the notation).

Then in [B2], Borchers used all this and new ideas, including his results on generalized Kac-Moody algebras, also called Borchers algebras, together with certain ideas from string theory, including the “physical space” of a bosonic string along with the “no-ghost theorem” of R. Brower, P. Goddard and C. Thorn [Br], [GT], to prove the remaining Conway-Norton conjectures for the structure  $V^\natural$ . What had remained to prove was that the formal series  $\sum (\text{tr } g|_{V_n^\natural}) q^n$  ((3) above, but now, rather, (3) for the known structure  $V^\natural$  instead of for a still-unknown space  $V$ ) for the Monster elements  $g$  (or conjugacy classes) not treated in [FLM2], [FLM5]—that is, the conjugacy classes outside the involution centralizer—were indeed the desired Hauptmoduls; the methods of [FLM2], [FLM5] did not handle these conjugacy classes. He accomplished this by constructing a copy of his “Monster Lie algebra” from the “physical space” associated with  $V^\natural$ , enlarged to a central-charge-26 vertex algebra closely related to the 26-dimensional bosonic-string structure mentioned above. He transported the known action of the Monster from  $V^\natural$  to this copy of the Monster Lie algebra, and by using his twisted denominator formula for this Lie algebra he proved certain recursion formulas for the coefficients of the formal series  $\sum (\text{tr } g|_{V_n^\natural}) q^n$  (that is, (3) for the known structure  $V^\natural$ ), for all Monster elements  $g$ . This entailed a generalization to Borchers algebras of the work [GL], which had generalized B. Kostant’s homology theorem [Ko]. The resulting recursion formulas for  $\sum (\text{tr } g|_{V_n^\natural}) q^n$  agreed with the “replication formulas” in [CN], satisfied by the coefficients of the Hauptmoduls listed in [CN]. By numerically verifying that the first few terms of some of the formal series  $\sum (\text{tr } g|_{V_n^\natural}) q^n$  (for  $g$  ranging through certain elements of the involution centralizer, whose action on  $V^\natural$  had been constructed in [FLM5]) agreed with the corresponding coefficients of the corresponding Hauptmoduls listed in [CN], he succeeded in concluding that all the graded traces  $\sum (\text{tr } g|_{V_n^\natural}) q^n$  for  $V^\natural$  must coincide with the formal series for the Hauptmoduls listed in [CN].

This remarkable work of Borchers has been further illuminated in a number of ways. In [Ju], E. Jurisich simplified Borchers's argument (proving the replication formulas for the structure  $V^\natural$ ) by exploiting a certain “large” free Lie algebra inside the Monster Lie algebra; this simplification is further discussed in [JLW]. Rather than a “Borel subalgebra” of the Monster Lie algebra, a certain natural “parabolic” subalgebra was used, allowing the simplification.

Moreover, in [CG], C. Cummins and T. Gannon discovered a conceptual proof that the replication formulas lead to the genus-zero property. In particular, the numerical checking in [B2] using the first few terms of the formal series constructed in [FLM2], [FLM5] can essentially be bypassed. Once one proves the replication formulas for the action of the Monster on  $V^\natural$  [B2] (or with the shorter argument in [Ju] (or [JLW])), then by [CG] one knows that the “McKay-Thompson series” for all the Monster elements acting on the structure  $V^\natural$  have the genus-zero property. Also, the fact [FLM2], [FLM5] that the graded dimension of  $V^\natural$  is the modular function  $J(q)$ , given certain established properties of the vertex operator algebra  $V^\natural$ , follows alternatively from a major theorem of Y. Zhu [Z] on the modular transformation properties of the graded dimensions of modules for suitable vertex operator algebras.

The original McKay-Thompson-Conway-Norton conjectures are conceptually proved. But there is also much, much more to monstrous moonshine (some of it mentioned below). See in particular Gannon's treatments in [Ga1] and [Ga2], which include references to many works and surveys.

As it turned out, then, the numerology of “monstrous moonshine” is much more than an astonishing relation between finite group theory and number theory; its underlying theme is the new theory of vertex (operator) algebras, itself the foundational structure for conformal field theory, which is in turn the foundational structure underlying string theory.

It's in this sense that (as I said at the beginning) vertex operator algebra theory is inherently “nonclassical” in the same way in which sporadic group theory and string theory are “nonclassical” in their respective domains.

In order to motivate the precise definition of vertex (operator) algebra, which I'll give later, I'll first repeat that there is a vertex operator algebra (namely,  $V^\natural$ ) whose symmetry group is the Monster  $\mathbb{M}$  and which implements the McKay-Thompson-Conway-Norton conjectures relating  $\mathbb{M}$  to modular functions including  $J(q)$ . But in fact, this vertex operator algebra  $V^\natural$  has the following three simply-stated properties—*properties that have nothing at all to do with the Monster*:

(1)  $V^\natural$ , which is an irreducible module for itself (proved in [FLM5]), is its only irreducible module, up to equivalence. C. Dong [D] proved this, and C. Dong-H. Li-G. Mason [DLM] proved the stronger result that every module for the vertex operator algebra  $V^\natural$  is completely reducible and is in particular a direct sum of copies of itself. *Thus the vertex operator algebra  $V^\natural$  has no more representation theory than does a field!* (I mean a field in the sense of mathematics, not physics. Given a field, every one of its modules—called vector spaces, of course—is completely reducible and is a direct sum of copies of itself.)

(2)  $\dim V_0^{\natural} = 0$ . This corresponds to the zero constant term of  $J(q)$ ; while the constant term of the classical modular function is essentially arbitrary, and is chosen to have certain values for certain classical number-theoretic purposes, the constant term must be chosen to be zero for the purposes of moonshine and the moonshine module vertex operator algebra.

(3) *The central charge of the canonical Virasoro algebra in  $V^{\natural}$  is 24.* “24” is the “same 24” so basic in number theory, modular function theory, etc. As mentioned above, this occurrence of 24 is also natural from the point of view of string theory.

*These three properties are actually “smallness” properties in the sense of conformal field theory and string theory. These properties allow one to say that  $V^{\natural}$  essentially defines the smallest possible nontrivial string theory* (cf. [Ha], [Na] and the Introduction in [FLM5]). (These “smallness” properties essentially amount to: “no nontrivial representation theory,” “no nontrivial gauge group,” i.e., “no continuous symmetry,” and “no nontrivial monodromy”; this last condition actually refers to both the first and third “smallness” properties.)

Conversely, *conjecturally* [FLM5],  $V^{\natural}$  is the unique vertex operator algebra with these three “smallness” properties (up to isomorphism). This conjecture turns out to be very hard to prove (without additional strong hypotheses); in any case, it remains unproved. It would be the conformal-field-theoretic analogue of the uniqueness of the Leech lattice in sphere-packing theory and of the uniqueness of the Golay code in error-correcting code theory, mentioned above. Proving this uniqueness conjecture can be thought of as the “zeroth step” in the program of classification of (reasonable classes of) conformal field theories. M. Tuite [Tu] has related this conjecture to the genus-zero property in the formulation of monstrous moonshine. With additional (strong) hypotheses assumed, uniqueness results have been proved by Dong-Griess-C. H. Lam [DGL] and by Lam-H. Yamauchi [LY].

Up to this conjecture, then, we have the following remarkable characterization of the largest sporadic finite simple group: *The Monster is the automorphism group of the smallest nontrivial string theory that nature allows, or more precisely, the automorphism group of the vertex operator algebra with the canonical “smallness” properties.* (As I mentioned above, space-time is 2-dimensional for this “toy-model” string theory. Bosonic 26-dimensional space-time is “compactified” on 24 dimensions, using the orbifold construction  $V^{\natural}$ ; again cf. [Ha], [Na] and the Introduction in [FLM5].) Note that (up to the conjecture) the “smallness” properties *characterize* the vertex operator algebra, but in order to actually *construct* it and to construct its automorphism group one needs the work in [FLM5] or the equivalent.

This definition of the Monster in terms of “smallness” properties of a vertex operator algebra provides a remarkable motivation for the definition of the precise notion of vertex (operator) algebra. The discovery of string theory (as a mathematical, even if not necessarily physical) structure sooner or later must lead naturally to the question of whether this “smallest” possible nontrivial vertex operator algebra  $V^{\natural}$  exists, and the question of what its symmetry group (which turns out to be the largest sporadic finite simple group) is. And on the



other hand, the classification of the the finite simple groups—a *mathematical problem of the absolutely purest possible sort*—leads naturally to the question of what natural structure the largest sporadic group is the symmetry group of; the answer entails the development of string theory and vertex operator algebra theory (and involves modular function theory and monstrous moonshine as well). The Monster, a singularly *exceptional* structure—in the same spirit that the Lie algebra  $E_8$  is “exceptional,” though  $\mathbb{M}$  is *far* more “exceptional” than  $E_8$ —helped lead to, and helps shape, the very *general* theory of vertex operator algebras. (The exceptional nature of structures such as  $E_8$ , the Golay code and the Leech lattice in fact played crucial roles in the construction of  $V^\natural$ , as is explained in detail in [FLM3] and [FLM5].)

Incidentally, whatever the ultimate role of string theory turns out to be in physics, string theory is here to stay; string theory has been “experimentally tested” very successfully—in mathematics (whether string theory is done by physicists or mathematicians or both), and in many, many ways, going far beyond what I have been discussing.

The results in [FLM5] include that  $V^\natural$  is defined over the field of real numbers, and in fact over the field of rational numbers, in such a way that the Monster preserves the real and in fact rational structure, and that the Monster preserves a rational-valued positive-definite symmetric bilinear form on this rational structure. More recent proofs that  $V^\natural$  is a vertex operator algebra have been found—by L. Dolan-P. Goddard-P. Montague [DGM], by Y.-Z. Huang [Hua3] and by M. Miyamoto [Mi]. (The proof in [FLM5] is perhaps still the shortest; any complete proof must include the full construction itself.) Huang’s proof of (the hard part of) the vertex-operator-algebra property of  $V^\natural$  uses the tensor product theory for modules for a (suitable) vertex operator algebra; I’ll mention this later. Y. Kawahigashi and R. Longo [KLo] have interpreted the “orbifold” construction of  $V^\natural$  in terms of algebraic quantum field theory, specifically, in terms of local conformal nets of von Neumann algebras on the circle.

The Monster is not the only sporadic finite simple group to which a vertex-operator-algebraic structure has been attached. G. Höhn [Ho] has constructed a vertex-operator-superalgebraic structure for the Baby Monster, which is involved in the Monster. Also, J. Duncan [Du1] has done so for the Conway group  $Co_1$ , and has proved the uniqueness of the structure. Evidence for the existence of such a structure was given in [FLM3]. See also Borchers-A. Ryba [BR]. In a remarkable development, Duncan ([Du2], [Du3]) has constructed two vertex-algebraic structures for a sporadic group *not* involved in the Monster, namely, the Rudvalis group, yielding moonshine-type phenomena, including a genus-zero property. This supports the hope, expressed in [FLM5], that all the sporadic groups (as well as all the other finite simple groups) can eventually be described in vertex-algebraic terms.

So, exactly what *is* a vertex operator algebra? And what *are* vertex operators? First of all, with what is now understood, *vertex operators are* (or rather correspond to) *elements of vertex operator algebras*, by analogy with how (for example) *vectors are elements of (abstract) vector spaces*; the notion of vector space is of course in turn defined by an axiom system. But before abstract

vector spaces had been formalized, vectors already “were” something (little arrows, etc.), and this of course helped motivate the eventual axiom system for the notion of vector space. Here is (an oversimplified version of) what vertex operators “already were”:

In string theory and conformal field theory, when two (closed) strings interact at a “vertex,” one has a standard picture that looks like a “pair of pants,” conformally equivalent to a three-punctured Riemann sphere (after a suitable interpretation). Such a picture is the string-theoretic analogue of a simple Feynman diagram that looks like the letter “Y”—a schematic diagram for two incoming particles interacting at a “*vertex*” and producing one outgoing particle. In the “pair of pants,” the *singularity* of the vertex in traditional (point-particle) quantum field theory is replaced by a *smooth* Riemann surface. This allows string theory to avoid the “ultraviolet divergences” in point-particle quantum field theory. In string theory and conformal field theory, such “Riemann-surface vertex diagrams” get “represented” by “vertex operators” acting on suitable infinite-dimensional vector spaces; vertex operators “describe” the particle (or rather, string) interactions in a given conformal field theory model. Geometric relations among Riemann-surface “diagrams” are reflected by algebraic and analytic relations among vertex operators.

I’ll next give a concrete example of a vertex operator, as it arose in mathematics:

For certain mathematical reasons, with R. Wilson [LW1] we focused on the “philosophical” problem of trying to construct the affine Kac-Moody Lie algebra

$$A_1^{(1)} = \widehat{\mathfrak{sl}(2)} = \mathfrak{sl}(2) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \quad (4)$$

(the “affinization” of the Lie algebra  $\mathfrak{sl}(2)$  of 2-by-2 matrices of trace 0), with Lie brackets given by

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + \text{tr}(ab)m\delta_{m+n,0}c$$

for  $a, b \in \mathfrak{sl}(2)$  and  $m, n \in \mathbb{Z}$  and

$$[c, \widehat{\mathfrak{sl}(2)}] = 0,$$

as some kind of “concrete” Lie algebra of (as-yet unknown) operators on some kind of “concrete” space.

In fact, our main reasons for formulating and trying to solve this problem were that we already knew some pieces of evidence, stemming from joint work with S. Milne [LM] and with A. Feingold [FL], that such a construction might eventually shed light on the classical Rogers-Ramanujan combinatorial identities. One of these two identities states that the number of partitions of a nonnegative integer  $n$  into parts congruent to 1 or 4 mod 5 equals the number of partitions of  $n$  into parts whose successive differences are at least 2, and the other of these identities states that the number of partitions of a nonnegative integer  $n$  into parts congruent to 2 or 3 mod 5 equals the number of partitions of  $n$  into parts whose successive differences are at least 2 and such that the

smallest part is at least 2. These two theorems have a long and interesting history and are highly nontrivial; cf. [A1]. When these two identities are written in their original, classical, generating-function form (cf. [A1]), each of them asserts the equality of two formal power series ( $q$ -series)—one of them a formal infinite product in  $q$  and the other a formal infinite sum. The work [LM], which used the Weyl-Kac character formula [Ka], showed that the product sides of the two Rogers-Ramanujan identities had something interesting to do with standard (= integrable highest weight)  $\widehat{\mathfrak{sl}(2)}$ -modules of levels 1 and 3; the “level” is the scalar by which the central element  $c$  in (4) acts. It seemed natural to try to “construct” these standard modules somehow, starting with the level 1 standard modules (the “basic” modules). The hope was to try to “discover” the sum sides of the Rogers-Ramanujan identities, somehow, in the level 3 standard modules. The Rogers-Ramanujan identities had been proved many times, but the question now was: What do the sum sides of the Rogers-Ramanujan identities “count,” in this new context? (Classically, they count partitions satisfying the difference-two condition.)

Here is the result of the work [LW1] (expressed in different, but equivalent, notation): Consider the (commutative associative) algebra

$$S = \mathbb{C}[y_{\frac{1}{2}}, y_{\frac{3}{2}}, y_{\frac{5}{2}}, \dots]$$

of polynomials in the formal variables  $y_n$ ,  $n = \frac{1}{2}, \frac{3}{2}, \dots$ . Form the expression

$$Y(x) = \exp \left( \sum_{n=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots} \frac{y_n}{n} x^n \right) \exp \left( -2 \sum_{n=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots} \frac{\partial}{\partial y_n} x^{-n} \right),$$

where “exp” is the formal exponential series and  $x$  is another formal variable commuting with the  $y_n$ ’s. The  $y_n$ ’s (understood as multiplication operators on  $S$ ) can be thought of as “creation operators” and the  $\frac{\partial}{\partial y_n}$ ’s as “annihilation operators,” acting on the “Fock space”  $S$ , using some terminology from quantum field theory. Together with the identity operator on the space  $S$ , they span an (infinite-dimensional) Heisenberg Lie algebra acting on  $S$ ; the commutators among these operators are the classical Heisenberg commutation relations, on infinitely many generators. The operator  $Y(x)$  is a well-defined formal differential operator in infinitely many formal variables, including the extra variable  $x$ . Viewing  $Y(x)$  as a generating function with respect to the formal variable  $x$ , we write

$$Y(x) = \sum_{j \in \frac{1}{2}\mathbb{Z}} A_j x^{-j},$$

thus giving a family of (well-defined) linear operators  $A_j$ ,  $j \in \frac{1}{2}\mathbb{Z}$ , acting on  $S$ . Each  $A_j$  can be computed, as a certain formal differential operator, in the form of an infinite sum of products of multiplication operators with partial differentiation operators, multiplied in this order; this infinite sum actually acts as a finite sum when applied to any given element of the space  $S$ . The explicit expression for each  $A_j$  is in fact complicated, and while one *can* write it down

explicitly, one does not want to have to do this, although in our original work we did in fact find these explicit formal differential operators  $A_j$  “directly”; it was only after the fact that we realized that if we added up *all* of these complicated operators  $A_j$  and thus formed their generating function as above, then all of these operators  $A_j$  could be described by the single product of exponentials  $Y(x)$ , which looked much simpler than any of the individual operators  $A_j$ .

The main point of this was:

**Theorem 1** [LW1] *The operators*

$$1, \ y_n, \ \frac{\partial}{\partial y_n} \left( n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \right) \text{ and } A_j \left( j \in \frac{1}{2}\mathbb{Z} \right)$$

*(1 is the identity operator on the space  $S$ ) span a Lie algebra of operators acting on  $S$ , that is, the commutator of any two of these operators is a (finite) linear combination of these operators, and this Lie algebra is a copy of the affine Lie algebra  $\widehat{\mathfrak{sl}(2)}$ .*

This operator  $Y(x)$  turned out to be a variant of the vertex operators that had arisen in string theory, as H. Garland pointed out (although that is not how we had found it). It turned out that vertex operators and symmetry are closely related. Instances of this had already been discovered in physics, including the works [H] and [BHN].

With what is now known, this operator  $Y(x)$  is an example of a “twisted vertex operator” (a vertex operator appearing in a twisted module for a vertex operator algebra). This particular vertex operator construction happens to be among the (many) ingredients playing crucial roles in the construction of  $V^{\mathfrak{h}}$ , as are the untwisted vertex operator constructions of I. Frenkel-V. Kac [FK] and G. Segal [S1] as well. These and other vertex operator constructions also enter into a variety of other, very different, mathematical problems. For instance, the operator  $Y(x)$  above was interpreted by E. Date, M. Kashiwara and T. Miwa [DKM] to be precisely the infinitesimal Bäcklund transformation for the Korteweg-de Vries hierarchy of differential equations in soliton theory; in Hirota’s bilinear formalism,  $Y(x)$  generates the multi-soliton solutions.

In Theorem 1, the  $\widehat{\mathfrak{sl}(2)}$ -module that is constructed using the twisted vertex operator  $Y(x)$  is one of the two basic (level 1 standard)  $\widehat{\mathfrak{sl}(2)}$ -modules. The other one of the two basic modules is constructed in exactly the same way, but with each operator  $A_j$  in Theorem 1 replaced by  $-A_j$  (or equivalently, with the generating-function  $Y(x)$  of the operators  $A_j$  replaced by its negative  $-Y(x)$ ).

Further work with Wilson led to the construction of structures we called “Z-algebras” [LW2]–[LW4], which provided a vertex-operator-theoretic interpretation and proof the Rogers-Ramanujan identities (mentioned above), in the following way (very briefly):

The (higher-level) standard  $\widehat{\mathfrak{sl}(2)}$ -modules of level  $k > 1$  can be constructed inside the tensor products of  $k$  copies of basic modules, but it was an open problem to construct these higher-level modules “concretely,” by exhibiting

natural bases of them. The work [LM] led to a natural conjecture that the Rogers-Ramanujan identities should “take place” inside the level 3 standard  $\widehat{\mathfrak{sl}(2)}$ -modules, with a structure that looks like a copy of the basic  $\widehat{\mathfrak{sl}(2)}$ -module somehow “factored out,” as a tensor factor. It turned out that the Heisenberg Lie subalgebra of  $\widehat{\mathfrak{sl}(2)}$  entering into Theorem 1 acts completely reducibly on each standard module  $L$ , and that the “vacuum space”  $\Omega_L$  in  $L$  for the action of this Heisenberg subalgebra (the subspace of  $L$  annihilated by all the “annihilation operators” mentioned above) has an easily-computed graded dimension. This vacuum space  $\Omega_L$  implemented the desired “factoring out,” and in case the level of  $L$  is 3, the  $q$ -series that are the graded dimensions of the spaces  $\Omega_L$  are exactly the product sides of the two Rogers-Ramanujan identities. The next problem appeared to be to find a basis of the vacuum space  $\Omega_L$  (for a level 3 standard module) that would exhibit the graded dimension of  $\Omega_L$  as the *sum* side of the corresponding Rogers-Ramanujan identity—the classical  $q$ -series generating function of the number of partitions of  $n$  satisfying one of the two difference-two conditions mentioned above.

Using the type of structure involved in Theorem 1, in [LW2]–[LW4] we eventually constructed certain operators that commute with the action of the Heisenberg Lie subalgebra of  $\widehat{\mathfrak{sl}(2)}$ , acting on suitable modules, and we called these operators “ $Z$ -operators” (“ $Z$ ” referring to the centralizing of this subalgebra). A typical  $Z$ -operator is a generating function of the shape

$$Z(x) = \sum_{j \in \mathbb{Z}} Z_j x^{-j},$$

where each  $Z_j$  is an operator of “degree”  $j$  that (because of the centralizing property) preserves  $\Omega_L$ . The goal seemed to be to prove that the vacuum space  $\Omega_L$ , for a level 3 standard module, has a basis of the form

$$Z_{j_1} Z_{j_2} \cdots Z_{j_n}, \quad j_i < 0, \tag{5}$$

with

$$j_1 \leq j_2 - 2, \quad j_2 \leq j_3 - 2, \dots, j_{n-1} \leq j_n - 2, \tag{6}$$

and with  $j_n \leq -2$  as well, in the case of the second of the Rogers-Ramanujan identities. (The subscripts  $j_i$  play the role of the negatives of the parts in a partition of a nonnegative integer.) This difference-two condition on the indices of such a basis monomial would exhibit the graded dimension of  $\Omega_L$  as the sum side of a Rogers-Ramanujan identity, and this would solve the problem. But achieving this would require the “straightening” of monomials (5) to obtain the inequalities (6) on the indices. Such “straightening” could be accomplished if there were good enough algebraic relations involving the generating function  $Z(x)$ , such as perhaps a commutator formula for  $[Z(x_1), Z(x_2)]$ .

But there *is* no such commutator formula. Instead, what turned out to be possible was the construction of “generalized commutation relations” of the shape

$$A(x_1, x_2)Z(x_1)Z(x_2) - B(x_1, x_2)Z(x_2)Z(x_1) = C(x_1, x_2), \tag{7}$$

where  $A(x_1, x_2)$  and  $B(x_2, x_1)$  are suitable formal expansions of suitable formal algebraic functions, and  $C(x_1, x_2)$  is some operator that is “simpler than” both  $Z(x_1)Z(x_2)$  and  $Z(x_2)Z(x_1)$ . The term “generalized” refers to the presence of the formal algebraic functions  $A$  and  $B$ ; in the case when these are 1, then one of course has ordinary commutation relations. Also, there are “generalized anticommutation relations,” of a generally still-more-complicated shape, that involve arbitrary numbers of generating functions  $Z(x_i)$  in general. *Each* of these generalized commutation and anticommutation relations can be viewed as the generating function of an *infinite family of relations* among monomials in the operators  $Z_j$ ; each such relation among such monomials is of the following form: A (well-defined) formal infinite linear combination of monomials in the  $Z_j$ ’s, with coefficients coming from the coefficients of the formal algebraic functions such as  $A$  and  $B$ , is equated with a “simpler” expression involving the  $Z_j$ ’s. (Again, if the formal algebraic functions  $A$  and  $B$  are 1 in (7), then each such relation among the  $Z_j$ ’s is a commutation relation of the following form:  $[Z_{j_1}, Z_{j_2}]$  equals a simpler expression such as perhaps a multiple of a single  $Z_j$  or a scalar.) These various types of generalized commutation and anticommutation relations entering into the solution of the present problem are detailed in [LW2]–[LW4]. What they were used for was to “straighten” monomials (5) to obtain the difference-two condition (6) on the subscripts, in the case of the the level 3 standard  $\widehat{\mathfrak{sl}(2)}$ -modules. This showed that the monomials (5) satisfying the difference-two condition *span* the vacuum space  $\Omega_L$ . Much more work was needed to prove that these difference-two monomials are also *linearly independent*, thus proving that they form a basis. Once these difference-two monomials were proved to form a basis, this vertex-operator interpretation and proof of the Rogers-Ramanujan identities was complete.

I had mentioned generalized commutation relations when I was discussing how certain such relations among certain vertex operators gave a natural “commutative affinization” of the (enlargement of) the Griess algebra acting on  $V^\natural$ , in [FLM2], [FLM5]. In that situation, the formal algebraic functions  $A$  and  $B$  in (7) are particularly (but deceptively!) simple:

$$A(x_1, x_2) = B(x_2, x_1) = x_1 - x_2. \quad (8)$$

That is, the relations yielding the commutative affinization of the 196884-dimensional enlargement  $\mathcal{B}$  of the Griess algebra in  $V^\natural$  are of the shape

$$(x_1 - x_2)[Z_1(x_1), Z_2(x_2)] = \text{simpler}, \quad (9)$$

where in this formula, the generating functions  $Z_i(x_j)$  now refer to the relevant vertex operators in [FLM2], [FLM5]. These relations, which we called “cross-bracket” relations in [FLM2], [FLM5], exhibit the desired commutative affinization, once one has constructed the space  $V^\natural$ .

The process discussed above for the level 3 standard  $\widehat{\mathfrak{sl}(2)}$ -modules was extended to all the standard  $\widehat{\mathfrak{sl}(2)}$ -modules in [LW2]–[LW4], and the result was a corresponding vertex-algebraic interpretation of a known family of generaliza-

tions of the Rogers-Ramanujan identities, which had been discovered by B. Gordon, G. Andrews and D. Bressoud. Many of these identities are treated in [A1]. The case of the level 2 standard  $\widehat{\mathfrak{sl}(2)}$ -modules actually gave certain infinite-dimensional Clifford algebras, and correspondingly, a “difference-one” condition, familiar in a natural basis of an exterior algebra, rather than a “difference-two condition.” It was for this reason that we thought of the general phenomenon, arising now for all the standard  $\widehat{\mathfrak{sl}(2)}$ -modules, as the emergence of a new “generalized Pauli exclusion principle” for a natural family of operators generalizing classical Clifford-algebra operators that are familiar in quantum mechanics and quantum field theory and producing fermionic particles; starting at level 3, fermionic (“difference-one”) statistics changed into “difference-two statistics,” and for the levels greater than 3, “difference-two-at-a-distance statistics,” reflecting the sum sides of the Gordon-Andrews-Bressoud identities. (See [LW2]–[LW4].)

However, we were unable to prove the linear independence of the relevant monomials, analogous to those (for the level-3 case) in (5), (6), for the levels greater than 3. This problem was solved by A. Meurman and M. Primc [MP1], who thus provided a vertex-algebraic proof of the Gordon-Andrews-Bressoud identities beyond the case of the Rogers-Ramanujan identities. Also, C. Husu [Hus1] discovered an elegant “multi-Jacobi-identity” interpretation and proof of the complicated generalized anticommutation relations in [LW2], [LW4].

The  $Z$ -algebra viewpoint in [LW2]–[LW4] was used by K. Misra ([Mis1]–[Mis4]), M. Mandia [Ma], C. Xie [X], S. Capparelli [Ca1] and M. Bos-K. Misra [BosMis] to construct difference-two-type bases for the vacuum spaces  $\Omega_L$  for certain standard modules for a range of affine Lie algebras, giving still more interpretations and proofs of the Rogers-Ramanujan and Gordon-Andrews-Bressoud identities, and Bos [Bos] proved that in fact the complete list of such occurrences of the Rogers-Ramanujan identities consists of certain low-level standard modules for the affine Kac-Moody algebras  $A_1^{(1)}$ ,  $A_2^{(1)}$ ,  $A_2^{(2)}$ ,  $A_7^{(2)}$ ,  $C_3^{(1)}$ ,  $F_4^{(1)}$  and  $G_2^{(1)}$ ; all these cases are covered by the papers mentioned.

One goal was to discover new identities with these ideas. In his Ph.D. thesis research at Rutgers, S. Capparelli set out to construct  $Z$ -algebra bases of the vacuum spaces  $\Omega_L$  for the standard  $A_2^{(2)}$ -modules of level 3, and he succeeded in obtaining a construction, and a pair of partition identities, but only as a conjecture, because he had not yet completed his  $Z$ -algebra proof of the linear independence of his spanning sets of  $Z$ -operator monomials in the vacuum spaces. (He completed this in [Ca2], and M. Tamba-C. Xie [TX] did so as well.) Meanwhile, in a talk at the Rademacher Centenary Conference in 1992, I mentioned Capparelli’s still-conjectured identities, or rather, one of them, in a survey of the research program. After confirming that the identity was indeed new, Andrews proved it, before the conference had ended (see [A2]), and soon afterward, K. Alladi, Andrews and Gordon formulated and proved a refinement and generalizations of it [AAG].

A central goal of these ideas was to find interesting new structure. The  $Z$ -algebra constructions mentioned so far are based on a certain twisting, the

same twisting as in [LW1]. In joint work with Primc [LP], we developed  $Z$ -algebras in the untwisted case, and using these and related ideas in this analogous but still surprisingly different setting, we constructed combinatorial bases exhibiting “difference-two-at-a-distance generalized fermionic statistics,” and corresponding (“fermionic”) character formulas, for the higher-level standard  $\widehat{\mathfrak{sl}(2)}$ -modules. In a sequel [MP3] to [MP1], Meurman and Primc discovered and exploited new structure related to [LP].

A. Zamolodchikov and V. Fateev introduced “nonlocal parafermion currents” in [ZF1] and a twisted analogue in [ZF2], and these conformal-field-theoretic constructions turned out to be essentially equivalent to the untwisted and twisted  $Z$ -algebra constructions in [LP] and [LW2], respectively; see [DL1], [DL2] for the untwisted case, and [Hus1] for the twisted case. The “nonlocality” in Zamolodchikov-Fateev’s terminology refers to the fact that in the notation above, the formal algebraic functions  $A(x_1, x_2)$  and  $B(x_1, x_2)$  in (7) are (formally) multiple-valued, and the term “parafermion” refers to the generalization of fermionic statistics mentioned above, or more precisely, to the reflection of such statistics in the form of the formal algebraic functions  $A$  and  $B$ . In the terminology of [DL2], the  $Z$ -algebra structures generate certain examples of “generalized vertex algebras” and “abelian intertwining algebras,” whose main axiom is a “generalized Jacobi identity,” incorporating the relevant formally-multiple-valued formal algebraic functions. Notions essentially equivalent to generalized vertex algebras were also introduced by A. Feingold-I. Frenkel-J. Ries [FFR] and G. Mossberg [Mos]. The introductory material in [DL2] includes discussions of these developments.

One of the fundamental methods classically used to study and prove the Rogers-Ramanujan identities was the Rogers-Ramanujan recursion, as explained in [A1]. But this recursion did not arise in any of the vertex-operator work on partition identities that I have mentioned. More recently, in joint work with Capparelli and A. Milas ([CapLM1], [CapLM2]), using vertex operator algebra theory in the context of “principal subspaces” ([FS1], [FS2]) of standard modules, we have been able to incorporate these into the theory, as well as the more general Rogers-Selberg recursions, satisfied by the sum sides of the Gordon-Andrews identities. This leads to new questions, being addressed in work of C. Calinescu [Cal1], [Cal2] and joint work with Calinescu and Milas [CalLM]. The main theme here is to use intertwining operators (which I’ll mention below) among modules for vertex (operator) algebras to find new structure. In fact, perhaps the main theme of all the work I’ve mentioned on partition identities is to use known and sometimes new identities as clues to look for interesting new structure.

Now I’ll give the definition of the notion of vertex operator algebra. The considerations I’ve mentioned (and related additional ones) led to the following variant ([FLM5] and [FHL]) of Borchers’s definition of the notion of vertex algebra; in this definition we use commuting independent formal variables  $x$ ,  $x_1$ ,  $x_2$ , etc.:



**Definition 1** A *vertex operator algebra* consists of a  $\mathbb{Z}$ -graded vector space

$$V = \bigoplus_{n \in \mathbb{Z}} V_{(n)}$$

(this grading, by *conformal weights*, is “shifted” from the grading we have already been using on  $V^\natural$ ; that grading is adapted to the modularity properties of the generating functions, including the  $J$ -function, that we have mentioned, while the present grading is adapted to the action of the Virasoro algebra on the vertex operator algebra, as we mention below) such that

$$\dim V_{(n)} < \infty \text{ for } n \in \mathbb{Z},$$

$$V_{(n)} = 0 \text{ for } n \text{ sufficiently negative,}$$

equipped with a linear map (the “*state-field correspondence*”)

$$\begin{aligned} V &\rightarrow (\text{End } V)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \end{aligned}$$

(where  $v_n \in \text{End } V$ , the algebra of linear operators on  $V$ ),  $Y(v, x)$  denoting the *vertex operator associated with*  $v$  (the letter “ $Y$ ” happens to look like the vertex Feynman diagram that we mentioned above), and equipped also with two distinguished homogeneous vectors  $\mathbf{1} \in V_{(0)}$  (the *vacuum vector*) and  $\omega \in V_{(2)}$  (the *conformal vector*). The axioms are: For  $u, v \in V$ ,

$$u_n v = 0 \text{ for } n \text{ sufficiently large}$$

(the *truncation condition*);

$$Y(\mathbf{1}, x) = \text{the identity operator on } V;$$

the *creation property*:

$$Y(v, x)\mathbf{1} \text{ has no pole in } x \text{ and its constant term is the vector } v$$

(which implies that the state-field correspondence is one-to-one); the *Virasoro algebra relations*:

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{m^3 - m}{12} \delta_{n+m, 0} c$$

for  $m, n \in \mathbb{Z}$ , where  $c \in \mathbb{C}$  (the *central charge*) and where

$$Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n) x^{-n-2}$$

(i.e., the conformal vector generates the Virasoro algebra);

$$L(0)v = nv$$

for  $n \in \mathbb{Z}$  and  $v \in V_{(n)}$ ;

$$\frac{d}{dx}Y(v, x) = Y(L(-1)v, x);$$

and finally, the main axiom (the vast bulk of the definition of vertex operator algebra), the *Jacobi identity*: Writing

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n, \quad (10)$$

the “formal delta function” (this really *is* a natural analogue of the Dirac delta function), we have:

$$\begin{aligned} & x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1) Y(v, x_2) \\ & \quad - x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) Y(v, x_2) Y(u, x_1) \\ & = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2), \end{aligned} \quad (11)$$

where each binomial, such as for example  $(x_1 - x_2)^{-n}$ , occurring in the expansions here is understood to be expanded in nonnegative integral powers of the *second* variable; the truncation condition insures that all the expressions here are well defined. The identity (11) is actually the generating function of an infinite list of identities among operators on  $V$ , obtained by equating the coefficients of all the monomials in  $x_0$ ,  $x_1$  and  $x_2$ .

The “physical” terms mentioned in this definition reflect the well-understood relation between this algebraic notion and the notion of “chiral algebra” in conformal field theory. The “language” of this definition, in particular, of the Jacobi identity, is “formal calculus,” which is discussed in detail in [FLM5], [FHL] and [LL], for instance. We first learned about the viewpoint of the formal delta function (10) from Garland; see also [DKM].

Formal calculus is completely different from the algebra of formal power series. Formal power series (with the powers of the formal variable(s) all non-negative), or formal Laurent series with the powers of the formal variable(s) truncated from below, form rings. They can be multiplied. But the formal series in the “formal calculus” of vertex operator algebra theory are “doubly infinite”; the powers of the formal variables in a given formal series can be arbitrary negative *or* positive integers, as is the case in (10) and (11), and sometimes even more generally, the powers of the formal variables in a given formal series must be allowed to be arbitrary rational or even *complex* numbers. These general kinds of “doubly infinite formal Laurent series” and so on certainly do *not* form rings. Yet there is a highly-developed algebra (“formal calculus”) for such formal series, and indeed, one needs this in vertex (operator) algebra theory and in particular in the representation theory (for instance, in the tensor product

theory for module categories for a suitable vertex operator algebra; I'll mention this later). A typical pattern in the theory is: First one does formal calculus, and then one (suitably and systematically) specializes the formal variables to complex variables and proves (analytic) convergence, for those results that need to be formulated analytically.

Now that we have the precise definition of the notion of vertex operator algebra, we have a canonical definition of the Monster without reference to finite group theory: It is the symmetry group of the (conjecturally) unique vertex operator algebra (a structure satisfying the Jacobi identity and the “relatively minor” axioms) having the three “smallness” properties mentioned above. Thus the Jacobi identity plus a few words determine the largest sporadic finite simple group.

Notice that there is a certain resemblance between the “shape” of the Jacobi identity (11) and the (schematic) generalized commutation relation (7); the three-variable formal-delta-function expressions in the left-hand side of (11) are analogous to the formal expressions  $A(x_1, x_2)$  and  $B(x_1, x_2)$  in (7). This is actually more than a coincidence. In [B1], Borchers had discovered identities equivalent, in retrospect, to certain consequences of (11). These included in particular a formula for the commutator  $[Y(u, x_1), Y(v, x_2)]$ , which can be obtained from (11) by taking the (formal) residue with respect to the formal variable  $x_0$  (that is, the coefficient of  $x_0^{-1}$ ) of each of the two sides of (11) and equating the results, as well as a formula for  $Y(Y(u, x_0)v, x_2)$ , which can be obtained by taking the residue with respect to the formal variable  $x_1$  of each of the two sides of (11) and equating the results. In [B1], these and other identities for a vertex algebra were actually expressed in component form (that is, in terms of the component operators  $v_n$  of the vertex operators  $Y(v, x)$ ) instead of generating-function form, but they could be recast in generating-function form. Also, Borchers used the second one of these two identities, together with certain other axioms, not including his commutator formula, is his definition of “vertex algebra.” When expressed, in retrospect, using the formal variables of (11), each of these two identities involves only two of the three formal variables  $x_0, x_1, x_2$  (the third variable being eliminated by the residue procedure).

Now, the identity of the shape (9) that was needed for the commutative affinization, using “cross-brackets,” in [FLM2], [FLM5] suggested the formulation of a *family* of generalized commutation relations, with the left-hand sides of the form

$$(x_1 - x_2)^n Z_1(x_1) Z_2(x_2) - (-x_2 + x_1)^n Z_2(x_2) Z_1(x_1) \quad (12)$$

for all  $n \in \mathbb{Z}$  (where the binomial expansion convention mentioned just after (11) is being used here), so that the case  $n = 0$  would give a commutator formula and the case  $n = 1$  would give an expression for the cross-bracket of type (9). In [FLM5] we were in fact able to do this, and the most natural way to formulate the resulting infinite family of identities was to put all these identities into a single generating function, using a new formal variable  $x_0$ ; that is, there was

now a formula for

$$\sum_{n \in \mathbb{Z}} x_0^{-n-1} (x_1 - x_2)^n Z_1(x_1) Z_2(x_2) - \sum_{n \in \mathbb{Z}} x_0^{-n-1} (-x_2 - x_1)^n Z_2(x_2) Z_1(x_1) \quad (13)$$

(where the operators called  $Z_i$  here are now the vertex operators entering into the construction of  $V^\natural$ ). *But (13) is exactly the left-hand side of (11)* (with the operators  $Z_i$  now playing the roles of the vertex operators in (11)). In other words, (11) is the generating function of an infinite family of generalized commutation relations. Also, we were able to prove in [FLM5] that this generating function actually equaled the right-hand side of (11), for all the vertex operators involved in  $V^\natural$ . Thus for us, in [FLM5] the idea for formulating, and proving, the Jacobi identity arose from the axioms and properties that Borchers wrote in [B1] together with the generalized-commutation-relation idea. Notice that the three expressions in (11) are analogous, and even quite similar, to one another. In fact in [FHL], an explicit three-variable symmetry of the Jacobi identity was formulated and analyzed. If one takes the residue with respect to any of the three formal variables in (11), the three-variable symmetry is “broken,” and correspondingly, with (11) not yet known to us, it seemed natural to try to construct the identity (11), completing this natural symmetry. The identity (11) exhibits “all” the information. It was for these reasons that we were initially thinking of (11) as the “master formula.” (As we said in the Introduction of [FLM5], Borchers informed us that he too found this identity, and in fact it is implicit in [B1]. It is also implicit in the work of physicists, as we have been discussing.)

But we decided instead in [FLM5] to call (11) the Jacobi identity because it is analogous to the Jacobi identity in the definition of Lie algebra: For  $u$  and  $v$  in a Lie algebra,

$$(\text{ad } u)(\text{ad } v) - (\text{ad } v)(\text{ad } u) = \text{ad}((\text{ad } u)v),$$

where  $\text{ad } u$  is the operation of left-bracketing with the element  $u$ . While the variables in (11) must be understood as formal and *not* complex variables (for instance, note that in the formal delta function (10) itself, the formal variable cannot be specialized to a complex variable; the resulting doubly-infinite formal series converges nowhere), it is in fact possible (and very important) to specialize the formal variables to complex variables in certain systematic, and subtle, ways. This process is crucial, in particular, for making the connection between the notion of vertex operator algebra and the notion of chiral algebra in conformal field theory, and it is also crucial for mathematical reasons. In fact, because of the fundamental complex-analytic geometry implicit in (11), suitably interpreted, we also called (11) the “Jacobi-Cauchy identity” in [FLM5], for reasons explained in detail in the Appendix of [FLM5].

The notion of vertex operator algebra, then, is indeed deeply analogous to the notion of Lie algebra, and *is actually the “one-complex-dimensional analogue” of the notion of Lie algebra* (which is the corresponding “one-real-dimensional” notion, in this sense); this statement can be made precise using the language and

viewpoint of operads, which I will mention below in connection with Huang’s work incorporating the geometry underlying the Virasoro algebra into the structure.

In addition to being the generating function of an infinite family of generalized-commutation-relation identities, (11), as we mentioned above, is the generating function of an infinite family of (generally highly-nontrivial) identities for the component operators  $v_n$  of the vertex operators in a vertex operator algebra  $V$ , one identity for each monomial in the three formal variables  $x_0$ ,  $x_1$  and  $x_2$ . Each identity in this infinite list involves binomial coefficients, coming from the three formal delta-function expressions. It is the *generating function form of these identities (namely, the Jacobi identity)* that is the natural analogue of the Jacobi identity in the definition of Lie algebra. Even more basically,  $Y(u, x)$  is itself the generating function of the infinite family of operators  $u_n$  acting on  $V$ , as we saw in the concrete example of the (twisted) vertex operator  $Y(x)$  above (acting on the space  $S$ , in that case; as we emphasized, the generating function is much easier to work with than the individual operators  $A_j$ ). Moreover, the single “ $x$ -parametrized product operation” taking the ordered pair  $(u, v)$ ,  $u, v \in V$ , to the generating function  $Y(u, x)v$  can certainly be thought of as specifying an infinite family of nonassociative product operations  $u_n v$  on  $V$ , for  $n \in \mathbb{Z}$ , corresponding to the powers of  $x$ . In very special cases, the “component identities” of the Jacobi identity include the relations defining affine Lie algebras; the Virasoro algebra; the infinite-dimensional “affinization” of the (modified) Griess algebra  $\mathcal{B}$ ; and a vast array of other remarkable algebraic structures. In “formal calculus,” it is generally much more natural, and much easier, to work with formal delta functions, and with generating functions in general, rather than with individual components of vertex operators and individual relations among them. Generating functions of otherwise very complicated objects, such as nonassociative product operations, or operators on a space, or identities among such operators, pervade vertex operator algebra theory and allow one to work efficiently.

There are many generalizations and analogues of (11), including twisted Jacobi identities, as in [FLM5]; generalized Jacobi identities for abelian intertwining algebras, etc., as in [DL1], [DL2], [FFR] and [Mos]; Huang’s much more subtle identity for *nonabelian* intertwining algebras [Hua5]; multi-Jacobi identities, as in [Hus1], [Hus2]; and “logarithmic analogues” of the Jacobi identity, both untwisted [L3] and twisted [DoyLM], which serve to “explain” and generalize work of S. Bloch [Blo] on the appearance of certain values of the Riemann zeta function that arose in certain vertex operator computations.

We have emphasized that the notion of vertex operator algebra is actually the “one-complex-dimensional analogue” of the notion of Lie algebra. But *at the same time that it is the “one-complex-dimensional analogue” of the notion of Lie algebra, the notion of vertex operator algebra is also the “one-complex-dimensional analogue” of the notion of commutative associative algebra* (which again is the corresponding “one-real-dimensional” notion). Again, operad language can be used to make this precise, as we comment below.

*The remarkable and paradoxical-sounding fact that the notion of vertex oper-*

ator algebra can be, and is, the “one-complex-dimensional analogue” of BOTH the notion of Lie algebra AND the notion of commutative associative algebra lies behind much of the richness of the whole theory, and of string theory and conformal field theory. When mathematicians realized a long time ago that complex analysis was qualitatively entirely different from real analysis (because of the uniqueness of analytic continuation, etc., etc.), a whole new point of view became possible. In vertex operator algebra theory and string theory, there is again a fundamental passage from “real” to “complex,” this time leading from the concepts of *both* Lie algebra and commutative associative algebra to the concept of vertex operator algebra and to its theory, and also leading from point particle theory to string theory.

This analogy with the notion of commutative associative algebra comes from the “commutativity” and “associativity” properties of the vertex operators  $Y(v, x)$  in a vertex operator algebra, detailed in [FLM5] and [FHL], and discussed in many places, including the book [LL]. These properties are rooted in conformal-field-theoretic properties of vertex operators, as in [BPZ]; see [Go]. In fact, the Jacobi identity (11) follows from the commutativity property, in the presence of certain “minor” axioms; see [FLM5], [FHL], [Go]. The term “commutativity” actually refers to a certain “commutativity of left-multiplication operations,” which is why it can (and in fact does) imply associativity and the Jacobi identity.

It is natural to ask: *Can the Jacobi identity axiom (11) in the definition of vertex (operator) algebra be simplified?* As we have discussed, the Jacobi identity is actually the generating function of an infinite list of generally highly-nontrivial identities, *and one needs many of these individual component identities in working with the theory.* But is there some “simpler” condition that in fact implies the Jacobi identity (in the presence of the “minor” axioms in the definition of vertex operator algebra)?

In fact there is, and this simpler condition, which is related to the “commutativity” property, does indeed look much simpler than the Jacobi identity axiom, but it turns out that the apparent simplicity is deceptive.

This simple-looking replacement axiom is:

For all  $u, v \in V$ , where  $V$  is a structure satisfying all the conditions in Definition 1 except the Jacobi identity (11), there exists a nonnegative integer  $k$  such that

$$(x_1 - x_2)^k [Y(u, x_1), Y(v, x_2)] = 0. \quad (14)$$

This “*weak commutativity*” condition, and also, more significantly, the theorem that it implies the Jacobi identity (in the presence of “minor” axioms), were discovered in [DL2], where (14) was actually a special case of a much more general condition, namely, the analogous assertion for generalized vertex algebras and abelian intertwining algebras, mentioned above. In that greater generality, formal algebraic functions schematically called  $A$  and  $B$  in (7) replace the expression  $(x_1 - x_2)^k$  in (14). All this is treated in [DL2], in the full generality. The special case (14) is discussed in the Introduction of [DL2], formula (1.4). The proof [DL2] that (14) implies the Jacobi identity, and that the generalizations

of (14) imply the corresponding generalized Jacobi identities, for generalized vertex algebras and abelian intertwining algebras, are continuations of the idea that “commutativity” implies the Jacobi identity. In [DL2], we were working with graded structures; just as a vertex operator algebra is  $\mathbb{Z}$ -graded, a generalized vertex algebra or abelian intertwining algebra is graded, too (actually  $\mathbb{Q}$ -graded), and this grading was useful in the proof [DL2] that weak commutativity and its generalizations imply the corresponding (generalized) Jacobi identities. Soon after [DL2], H. Li was able to remove the grading hypothesis, and in particular, he was able to prove that for a *vertex algebra* (without grading), weak commutativity (14) implies the Jacobi identity, in the presence of “minor” axioms. This and a number of related results are covered in [LL].

Notice that the condition (14) is reminiscent of the generalized commutation relations discussed above, such as (7) and in particular, (9); in fact, (14) is of course an example of a generalized commutation relation.

The fact that such a simple-looking generalized commutation relation as (14) can serve as an axiom replacing the Jacobi identity in the definition of vertex operator algebra is not as useful as it might seem. In fact, starting in [DL2] itself, we chose not to take (14) as an “official” replacement axiom, in spite of the fact that we proved, and stated, there that it could be taken as a replacement axiom. There are essentially three reasons why we have chosen not to take (14) as an “official” axiom replacing the Jacobi identity: First, one needs “all” the information in the Jacobi identity (and in the relevant generalized Jacobi identities, in the context of generalizations of the notion of vertex operator algebra). Second, if one wants to prove that a certain structure  $V$  is indeed a vertex (operator) algebra, it is essentially just as hard to prove the condition (14) as it is to prove the Jacobi identity, for all  $u, v \in V$ . (In other words, the proof that (14) implies the Jacobi identity is quite short, and in particular is much simpler than the proof that either (14) or the Jacobi identity holds for all  $u, v \in V$ , for interesting examples of vertex (operator) algebras.) And third, (14) fails as a replacement axiom for the notion of *module* for a vertex (operator) algebra. In general, one can think of a module for an algebraic structure as a space on which the algebra acts linearly, such that all the axioms (in the definition of algebra) that make sense hold; this principle is compatible with the standard definition of “module” for a Lie algebra and the standard definition of “module” for an associative algebra, for instance. This principle indeed motivates the standard definition of “module” for a vertex (operator) algebra, and using (14) in place of the Jacobi identity would *not* lead to the correct notion of module. Analogous comments hold in generalized settings such as abelian intertwining algebras, and also, twisted modules. For instance, in [FLM5], certain twisted Jacobi identities were proved, and these identities endowed certain “twisted” spaces with what came to be called twisted module structure (although the term “twisted module” was not used on [FLM5]); this twisted module structure was necessary for the construction of  $V^{\natural}$ . (I mentioned above that the vertex operator  $Y(x)$  entering into Theorem 1 is an example of a “twisted vertex operator,” that is, a vertex operator appearing in a twisted module.)

As we have been suggesting, in vertex operator algebra theory it is notoriously difficult to construct nontrivial *examples* of vertex operator algebras, even examples that are much simpler than  $V^{\natural}$ , and of course one cannot do the theory without examples; in fact, the theory is so rich because the examples are so rich.

*How can one efficiently construct families of examples of vertex operator algebras and their modules?*

In classical algebraic subjects like group theory, Lie algebra theory, etc., one of course has many interesting examples available from the beginning, guiding the development of the general theory. In vertex operator algebra theory, *there are no essentially nontrivial examples that are easy to construct and prove the axioms for*. (The only “easy” examples of vertex algebras are commutative associative algebras equipped with derivations; see [B1].) This is yet another reason why vertex operator algebra theory is inherently “nonclassical.” In “classical” mathematics, there simply *were no nontrivial examples of vertex operator algebras* “lying around waiting to be axiomatized,” in contrast with, say, vector spaces, groups, Lie algebras, etc., etc.

A conceptually elegant, extremely general and extremely convenient-to-use solution of the problem of constructing families of examples of vertex operator algebras, and also modules for them, was developed by H. Li ([Li1], [Li2]), generalizing earlier constructions of examples of vertex operator algebras, including, among others, constructions of B. Feigin-E. Frenkel [FF] and I. Frenkel-Y. Zhu [FZ] (and constructions in [FLM5]). Briefly, Li formulated a subtle notion of *representation of*, as opposed to *module for*, a vertex operator algebra, by developing the theory of a “vertex-algebraic analogue” of the notion of the usual endomorphism algebra  $\text{End } W$  of a vector space  $W$  and by defining a *representation of a vertex algebra*  $V$  on  $W$  to be a (suitable kind of) homomorphism from  $V$  to this endomorphism-algebra structure on  $W$ . This endomorphism-algebra structure has roots in conformal field theory. It has also been exploited in work of B. Lian and G. Zuckerman [LZ1], [LZ2]. Li’s analysis of this point of view culminated in relatively-easy-to-implement sufficient conditions for constructing families of vertex operator algebras *and* their modules. An enhanced treatment of this work of Li’s is the main goal of the book [LL], which also highlights general theorems of E. Frenkel-V. Kac-A. Radul-W. Wang [FKRW], Meurman-Primc [MP2] and X. Xu [Xu] useful for constructing families of examples of vertex (operator) algebras, including those based on the Virasoro algebra, those based on Heisenberg Lie algebras, those based on affine Lie algebras, and those based on lattices. (It happens, though, that such theorems do not serve to simplify the construction of the vertex operator algebra  $V^{\natural}$ , even though all the kinds of vertex operator algebras just mentioned do enter into the construction of  $V^{\natural}$ .)

The analogy between the notion of vertex operator algebra and the notion of commutative associative algebra is in fact directly related to the conformal-field-theory viewpoint. This analogy was pointed out by I. Frenkel [F], in the initiation of a program to construct (geometric) conformal field theory using vertex operator algebras. In [Hua1], [Hua4], Huang introduced a precise, and



deep, analytic-geometric notion of “geometric vertex operator algebra,” and established that it is equivalent to the (algebraic) notion of vertex operator algebra. Formulating and proving the geometric aspects of the action of the Virasoro algebra was the hard part, involving differential geometry and analysis on infinite-dimensional moduli spaces; the infinite-dimensionality comes from the arbitrariness of analytic local coordinates at punctures on a Riemann sphere. The sewing of multipunctured Riemann spheres, with analytic local coordinates vanishing at the punctures (both the “incoming” and “outgoing” punctures), is reflected in the structure of vertex-operator-algebraic operations. In particular, the *formal* variables are systematically specialized to *complex* variables, and one does extensive analysis in addition to extensive algebra.

An interpretation—actually, restatement—of these constructions and theorems of Huang, including some discussion of the principle mentioned above that the notion of vertex operator algebra is a natural “complexification” of the notion of both Lie algebra and of commutative associative algebra, appears in [HL1], [HL2], in the language of operads. What is being “complexified” is the (one-real-dimensional) operads underlying both the notion of Lie algebra and the notion of commutative associative algebra; the new analytic partial operad underlying the notion of vertex operator algebra [Hua4] (cf. [HL1], [HL2]) is one-complex-dimensional. In fact, this one-complex-dimensional partial operad “dictates” the algebra of vertex operator algebra theory. More precisely, when one takes into account the arbitrary local coordinates vanishing at the punctures, one really has an infinite-dimensional structure. The algebraic operations are “mediated” by the infinite-dimensional analytic geometry of this partial operad, in a precise way; a vertex operator algebra becomes a *representation* of this analytic partial operad. Again, we are seeing a mathematical reflection of a passage from point particle theory to string theory.

In an extensive series of works, K. Barron ([Ba1]–[Ba6]) has carried out a sophisticated super-geometric (super-conformal-field-theoretic) analogue of this work of Huang’s, using super-Riemann spheres with general superconformal local coordinates vanishing at the “incoming” and “outgoing” punctures, and using vertex operator superalgebras endowed with the appropriate super-geometric structure.

As we have mentioned a number of times, in vertex operator algebra theory, it is crucial to make precise distinctions between formal variables and complex variables. This distinction is particularly dramatic in this work of Huang and Barron. First they had to carry out elaborate formal algebra, and then they had to systematically specialize the (infinitely many) formal variables to complex variables to obtain the desired results. In classical mathematics, one is used to being (appropriately!) “careless” about the distinction between formal and complex variables; for instance, one routinely writes formal expressions such as  $\sum_{n \geq 0} x^n$  without saying much about whether the variable  $x$  is supposed to be formal or complex. In this simple example, it is so clearly understood that this geometric series converges for certain complex numbers  $x$  and not others that the notation  $\sum_{n \geq 0} x^n$  can easily be used, in the same discussion, for either the formal sum or the convergent series, depending on what is being said.

However, in vertex operator algebra theory, where one also needs both formal and complex variables, the formal algebra (and “formal calculus,” mentioned above in connection with (11), in which the three formal variables *cannot* all be specialized to complex variables) becomes so subtle and elaborate that it is necessary to be very explicit about the distinction between the two kinds of variables. Correspondingly, in Huang’s and Barron’s work, there are many formal theorems, which cannot be initially and directly formulated in terms of complex variables, and these theorems are then systematically applied to give analytic consequences. One particular “formal” theorem that was proved by Huang and extended to the superalgebraic setting by Barron has been considerably generalized in [BHL], to a factorization theorem for formal exponentials, in a setting involving arbitrary infinite-dimensional Lie algebras. This theorem reflects and generalizes the formal algebra required for the sewing of Riemann spheres or super-Riemann spheres with general coordinates vanishing at punctures that are sewn together.

Huang’s work on geometric vertex operator algebras is a major step of many in a program to use vertex operator algebra theory and its representation theory to construct conformal field theories in the precise sense of G. Segal’s and M. Kontsevich’s definition of the (mathematical) notion of conformal field theory (see [S2]–[S4]); this definition was clarified by P. Hu–I. Kriz ([HuK1], [HuK2]). Geometric vertex operator algebras amount to the holomorphic, genus-zero part of the construction of conformal field theories. The term “genus-zero” now refers to the Riemann spheres mentioned above; it does not refer to the “genus-zero property” of the discrete subgroups of  $SL(2, \mathbb{R})$  discussed earlier in connection with moonshine. This use of “genus zero” corresponds to conformal field theory at “tree level,” in physics terminology.

A major achievement of this program so far is Huang’s proof ([Hua7]–[Hua11]), in a general setting, of the Verlinde conjecture and his solution of the problem of constructing modular tensor categories from the representation theory of vertex operator algebras. E. Verlinde conjectured [Ve] that certain matrices formed by numbers called the “fusion rules” in a “rational” conformal field theory are diagonalized by the matrix given by a certain natural action of the fundamental modular transformation  $\tau \mapsto -1/\tau$  on “characters.” A great deal of progress has been achieved in interpreting and proving Verlinde’s (physical) conjecture and the related “Verlinde formula” in mathematical settings, in the case of the Wess–Zumino–Novikov–Witten models in conformal field theory, which are based on affine Lie algebras. On the other hand, G. Moore and N. Seiberg [MSe1], [MSe2] showed, on a physical level of rigor, that the *general* form of the Verlinde conjecture is a consequence of the axioms for rational conformal field theories, thereby providing a conceptual understanding of the conjecture. In the process, they formulated a conformal-field-theoretic analogue, later termed “modular tensor category” by I. Frenkel, of the classical notion of tensor category for representations of (i.e., modules for) a group or a Lie algebra. A modular tensor category is in particular a braided tensor category that is also rigid and “nondegenerate.”

Now, there is a general tensor product theory for modules for a suitably

general vertex operator algebra, a theory based on intertwining operators [FHL] among modules (the dimensions of spaces of intertwining operators are the “fusion rules” mentioned above): The tensor product functors and appropriate structure were constructed in [HL4]–[HL6], and in [Hua2] Huang proved a general operator-product-expansion theorem for intertwining operators, enabling him to construct the natural associativity isomorphisms between suitable tensor products of triples of modules. (The existence of such an operator product expansion was a key *assumption*—not *theorem*—in [MSe2].) The resulting braided tensor category structure was enhanced in [HL3] to what we called “vertex tensor category” structure. This structure is much richer than braided tensor category structure. Vertex tensor category structure is “mediated” by the analytic partial operad [Hua4], [HL1], [HL2], based on multipunctured Riemann spheres with arbitrary analytic local coordinates vanishing at the punctures, by analogy with how the structure of ordinary, classical, braided tensor categories is “mediated” by operadic structure in one real dimension. That is, not only is the concept of vertex operator algebra *itself* “based” on the one-complex-dimensional operadic structure discussed above, but so is a *tensor category theory* [HL3] of modules for a (suitable) general vertex operator algebra. In particular, in place of a single tensor product functor, there is a natural *family* of tensor product functors, indexed by a power of the determinant line bundle over the moduli space of three-punctured Riemann spheres with analytic local coordinates vanishing at the punctures, and the natural associativity isomorphisms among triple tensor products, and the coherence, are controlled by this geometric structure; this is explained in [HL3]. In this tensor product theory, the underlying vector space of the tensor product of suitable modules for a vertex operator algebra is *not* the tensor product vector space of the modules. Instead, intertwining operators among triples of modules form the starting point for a family of analytically-defined tensor product functors, and it is a subtle matter to construct the tensor product spaces.

The work [HL3]–[HL6] and [Hua2] was originally inspired by the work of D. Kazhdan and G. Lusztig, starting in [KLu1]–[KLu3], constructing a tensor product theory for certain categories of modules of a fixed non-positive-integral level for an affine Lie algebra. However, while the theory of [HL3]–[HL6] and [Hua2] applies to the module categories of many families of vertex operator algebras, this theory does not include [KLu1]–[KLu3] as a special case, because the modules considered by Kazhdan-Lusztig are not semisimple. But recently, in joint work with L. Zhang [HLZ], we have generalized the tensor product theory [HL4]–[HL6], [Hua2] to “logarithmic” tensor product theory, which indeed accommodates suitable non-semisimple module categories. It turned out the the work necessary for this generalization was considerable. In particular, the already-intricate formal calculus necessary for [HL3]–[HL6] and [Hua2] had to be extended to “logarithmic formal calculus,” and many of the arguments in [HL4]–[HL6], [Hua2] had to be replaced by new ones, for the generalization. Using the work [HLZ], Zhang has succeeded [Zha] in recovering the braided tensor category of Kazhdan-Lusztig as a special case of the theory, and of endowing it with vertex tensor category structure.

To return to Huang's work announced in [Hua7]— Under very general and natural conditions on a simple vertex operator algebra  $V$ , Huang proved the Verlinde conjecture, and using this result, he proved the rigidity and in fact modularity of the braided tensor category constructed in [HL4]–[HL6], [Hua2]. Zhu's theorem [Z], which I mentioned above, on modular transformation properties of the graded dimensions of modules, is necessary in the formulation of the Verlinde conjecture in this general setting, and Zhu's method of proof plays an important role in the development of the genus-one theory. In Huang's proof of the Verlinde conjecture, a crucial step was to prove the modular invariance of spaces of multipoint correlation functions involving compositions of (multivalued) intertwining operators (as opposed to single-valued vertex operators). For this, Huang had to develop a new, analytically-based method [Hua9], since Zhu's method (mainly, his use of the commutativity property of vertex operators) cannot be generalized to this multivalued setting. (This situation is actually somewhat analogous to the situation discussed above in connection with generalized commutation relations (7), where it was impossible to construct *ordinary* commutation relations, which would have been easier to use, if they had existed. In Huang's proof of the Verlinde conjecture, however, the situation is even more subtle.) Huang established natural duality and modular invariance properties for genus-zero and genus-one multipoint correlation functions constructed from intertwining operators for a vertex operator algebra satisfying general hypotheses, and as I have mentioned, the multiple-valuedness of the multipoint correlation functions led to considerable subtleties that had to be handled analytically and geometrically, rather than just algebraically. Then, with these results having been established, the strategy of Huang's proof of the Verlinde conjecture reflected the pattern of [MSe1], [MSe2]; he used these results to establish two formulas that Moore and Seiberg had derived from strong assumptions, namely, the axioms for rational conformal field theory, which of course cannot be assumed here. It is much harder to (mathematically) prove the axioms in [S2]–[S4] or the axioms in [MSe2] for conformal field theory than it is to prove the Verlinde conjecture; in Huang's work, the truth of Verlinde conjecture was needed to *prove* the desired properties of the tensor category. In turn, his proof of the general Verlinde conjecture required a large amount of (existing and new) representation theory of vertex operator algebras. The hypotheses of the theorems entering into Huang's work are very general, natural and purely algebraic, and have been verified in a wide range of important examples, while the theory itself is heavily analytic and geometric as well as algebraic.

In another important direction, a great deal of extensive and deep work has been done in conformal field theory using algebraic geometry. I will only mention the books of A. Beilinson and V. Drinfeld [BD], and of E. Frenkel and D. Ben-Zvi [FB]. The review [Hua6] of [FB] includes an interesting discussion of the relation between the algebro-geometric viewpoint and the viewpoint of the representation theory of vertex operator algebras.

I would like to mention a certain inspiring article by M. Atiyah [A], in which among many other things he compares geometry and algebra; discusses the interaction between mathematics and physics; and comments on string the-

ory and also on finite simple groups, and in particular the Monster and its connections with elliptic modular functions, theoretical physics and quantum field theory—some of which we’ve actually been discussing. Among his many stimulating comments are these, about “the dichotomy between geometry and algebra”: “Geometry is, of course, about space... Algebra, on the other hand...is concerned essentially with time. Whatever kind of algebra you are doing, a sequence of operations is performed one after the other, and ‘one after the other’ means you have got to have time... Algebra is concerned with manipulation in *time*, and geometry is concerned with *space*. These are two orthogonal aspects of the world, and they represent two different points of view in mathematics. Thus the argument or dialogue between mathematicians in the past about the relative importance of geometry and algebra represents something very fundamental.”

In the spirit of what we’ve been discussing: While a string sweeps out a two-dimensional (or, as we’ve been mentioning, one-complex-dimensional) “world-sheet” in space-time, a point particle of course sweeps out a one-real-dimensional “world-line” in space-time, with time playing the role of the “one real dimension,” and this “one real dimension” is related in spirit to the “one real dimension” of the classical operads that I’ve briefly referred to—the classical operads “mediating” the notion of associative algebra and also the notion of Lie algebra (and indeed, any “classical” algebraic notion), and in addition “mediating” the classical notion of braided tensor category. The “sequence of operations performed one after the other” is related (not perfectly, but at least in spirit) to the ordering (“time-ordering”) of the real line. But as we have emphasized, the “algebra” of vertex operator algebra theory and also of its representation theory (vertex tensor categories, etc.) is “mediated” by an (essentially) one-*complex*-dimensional (analytic partial) operad (or more precisely, as we have mentioned, the infinite-dimensional analytic structure built on this). When one needs to compose vertex operators, or more generally, intertwining operators, after the formal variables are specialized to complex variables, one must choose not merely a (time-)ordered sequencing of them, but instead, a suitable complex number, or more generally, an analytic local coordinate as well, for each of the vertex operators. This process, very familiar in string theory and conformal field theory, is a reflection of how the one-complex-dimensional operadic structure “mediates” the algebraic operations in vertex operator algebra theory. Correspondingly, “algebraic” operations in this theory are not intrinsically “time-ordered”; they are instead controlled intrinsically by the one-complex-dimensional operadic structure. The “algebra” becomes intrinsically geometric. “Time,” or more precisely, as we discussed above, the one-real-dimensional world-line, is being replaced by a one-complex-dimensional world-sheet. This is the case, too, for the vertex tensor category structure on suitable module categories. In vertex operator algebra theory, “algebra” is more concerned with one-complex-dimensional geometry than with one-real-dimensional time.

As we have discussed, the Monster is indeed (very deeply) related to string theory. I mentioned above that one of the distinguishing features of the moonshine module vertex operator algebra  $V^\natural$  is that its representation theory is trivial. This means in particular that the braided tensor category attached to

it is also trivial. But its *vertex* tensor category, and correspondingly, its “one-complex-dimensional world-sheet” algebra, is *not* trivial. With Huang, we have been thinking about whether it will be possible to realize the Monster geometrically, in terms of the complex operad.

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